Resit Analysis on Manifolds WBMA013–05

Tuesday 09.04.2023, 15:00 – 17:00

You are allowed to use your cheatsheet (handwritten A4 page) during the exam. Make sure to clearly explain the steps in your proofs and computations. The exam consists of two pages with a total of 4 exercises. You get 10 points for free.

Exercise 1. (16 points)

Consider the map $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$\Phi(x, y, z, w) = \left(x^2 + y, x^2 + y^2 + z^2 + w^2 + y\right) .$$

Show that $S = \{p \in \mathbb{R}^4 \mid \Phi(p) = (0, 1)\}$ defines a smooth manifold. What is its dimension? Justify your answer.

Exercise 2. (5 + 6 + 8 + 5 = 24 points)

Let (x, y) denote the standard coordinates on $M = \mathbb{R}^2$. This example shows that coordinate vectors in the tangent space depend on the whole coordinate system and not just on the single coordinate function they are associated to.

1. Show that $(\widetilde{x}, \widetilde{y})$, where $\widetilde{x} = x$ and $\widetilde{y} = x^3 + y$ are global smooth coordinates on M.
2. How are $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial \widetilde{x}}$ defined? Is it clear that they are sections of TM? Justify your answer.
3. Compute $\frac{\partial}{\partial \widetilde{x}}$ and $\frac{\partial}{\partial \widetilde{y}}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.
4. To prove the claim, find a point $p \in M$ such that $\frac{\partial}{\partial x}\Big|_p \neq \frac{\partial}{\partial \widetilde{x}}\Big|_p$ or $\frac{\partial}{\partial y}\Big|_p \neq \frac{\partial}{\partial \widetilde{y}}\Big|_p$.

Exercise 3. (8 + 7 + 8 + 5 = 28 points)

Let $M = \mathbb{R}^3$, $\eta := dz - \frac{1}{2}(xdy - ydx) \in \Omega^1(M)$ and $X, Y, Z \in \mathfrak{X}(\mathbb{R}^3)$ defined by $X := \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$, $Y := \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$ and $Z := \frac{\partial}{\partial z}$.

1. Show that $\mathcal{D} = \ker(\eta)$ is a subbundle of $T\mathbb{R}^3$.
   Hint: if you are really not sure what to do, you can proceed as follows. For $q \in M$, denote $\mathcal{D}_q := \ker(\eta_q) \subset T_qM$. For all $q \in M$, show that $\mathcal{D}_q$ is closed under addition and multiplication, and contains 0_q.
2. Show that $\{X, Y\}$ is a frame for $\mathcal{D}$;
3. Compute $[X, Y]$;
4. Is the vector field $[X, Y]$ a section of $\mathcal{D}$? Justify your answer.

Exercise 4 (8 + 8 + 6 = 22 points)

In this exercise we keep using the same notation of Exercise 3 but can be solved independentlyn of that. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection on the (x, y)-plane, that is, $\pi(x, y, z) = (x, y)$. It can be assumed that $\eta \wedge d\eta$ defines a volume form on M.

Let $\gamma = (\gamma^1, \gamma^2, \gamma^3) : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth curve in M such that the following holds:
(a) for all $t \in I$, $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$;
(b) $\pi \circ \gamma = (\gamma^1, \gamma^2)$ is a closed (possibly self-intersecting) planar curve.

1. Show that $\int_\gamma \frac{1}{2}(xdy - ydx)$ coincides with the signed area of the region $\Omega \subset \mathbb{R}^2$ bounded by the curve $\pi \circ \gamma$;
   Hint: what is the planar integral to compute the area of Ω?
2. Show that condition (a) in the definition of γ implies that $\gamma^3(t)$ is proportional to the signed area of the region Ω bounded by $\pi \circ \gamma$ on the (x, y)-plane;
   Hint: compute $\eta_{\gamma(t)}(\dot{\gamma}(t)) = 0$ and integrate the resulting expression.
3. Show that γ is a closed curve in M if and only the signed area of Ω vanishes.